Supersymmetry on Lattice Using Ginsparg-Wilson Relation

Hiroto. So[a]Niigata]Department of Physics, Niigata University, Ikarashi 2-8050, Niigata, 950-2181, Japan, Naoya Ukita[b][Niigata]*

[a][

The Ginsparg-Wilson(G-W) relation for chiral symmetry is extended for a supersymmetrical(SUSY) case on a lattice. It is possible to define exact lattice supersymmetry which are devided into two different cases according to using difference operators. $U(1)_R$ symmetry on the lattice is also realized as one of exact symmetries. For an application, the extended G-W relation is given for a two-dimensional model with chiral multiplets.

1. Introduction

Although numerical QCD studies have succeeded in computer simulations by the lattice framework, the problem of chiral fermion[1] is remained. Kaplan's domain wall fermion[2] and the overlap formula[3] as the hamiltonian formalism is a nice idea if the chiral charge is sufficiently separate because the total theory is vector-like and consistent with the no-go theorem[1]. Recently, Neuberger[4] found that the Dirac operator by his overlap formula is satisfied with the Ginsparg-Wilson(G-W) relation[5]. Furthermore, although the Dirac operator is not chiral-invariant, the new symmetry found by Lüscher [6] is a modified version of chiral symmetry. It is noted that the usual path-integral measure is not invariant under the new symmetry and the chiral anomaly is generated.

Lattice formulations of supersymmetry(SUSY) were attempted by several authors[7] - [10]. But there are many problems such as definition of SUSY on lattice, non-locality and breakdown of Leibniz rule, because the SUSY tansformation need not only chiral fermions but also derivative operators. In this article, we explain a SUSY-extended G-W relation[11]. A new symmetry for lattice SUSY can be defined without any ambiguities in replacing derivative operators with difference ones. As the compensation, it is necessary for the block-spin funcion constructing a lattice theory. As a example, a SUSY free action in two-dimension is constructed. In addition to fermionic and bosonic G-W relations used by Aoyama-Kikukawa[9], we can derive the further two relations among fermions and bosons for the lattice action.

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2. Derivation of SUSY Ginsparg-Wilson relation

The original G-W relation was derived as an identity for the Gaussian type effective action. It should be recognized as a kind of Ward-Takahashi identity for the chiral transformation of the Gaussian type effective action and as the property of the lattice theory near the continuum limit with a chiral symmetry[12].

Due to the no-go theorem[1], we must prepare two chiral matter fields. A SUSY transformations for two chiral-multiplets, $\Phi_j = (\phi_j, \psi_j, F_j)^{\mathrm{T}}$ $j = 1, 2$ in the continuumm theory is defined as

$$\delta_\epsilon \Phi_j = Q(\epsilon, \bar{\epsilon})\Phi_j \tag{1}$$

$$\delta_\epsilon \bar{\Phi}_j = \bar{\Phi}_j \bar{Q}(\epsilon, \bar{\epsilon}) \tag{2}$$

where an Euclidean space is considered because we construct the Euclidean lattice theory. The dependency on index j is trivial and we suppress it from now. We begin with a continuum theory, and define its regularized theory on a cubic lattice by performing a block-spin transformation. A lattice point is expressed by an integer vector, $\{n_\mu a\}$, where a is lattice constant. We take for simplicity $a = 1$. The block-spin transformation from $\Phi(x)$ to Φ_n is given by

$$\Phi_n \sim \int \mathrm{d}x f_n(x)\Phi(x) \quad \equiv \quad \langle f_n, \Phi\rangle \tag{3}$$

$$\bar{\Phi}_n \sim \int \mathrm{d}x f_n(x)\bar{\Phi}(x) \quad \equiv \quad \langle f_n, \bar{\Phi}\rangle \tag{4}$$

where $f_n(x)$ is a block-spin function with finite support around $x_\mu = n_\mu$, and co-moving with $x_\mu - n_\mu$: $f_n(x) = f(x-n)$. $\langle , \rangle$ implies the usual inner product in a function space.

According to Ginsparg and Wilson,[5] we may define a Gaussian effective action $A_{\rm eff}$ by using a SUSY action A_c in the continuum theory*:

$$\begin{aligned} &\exp(-A_{\rm eff}[\Psi_n, \bar{\Psi}_n]) \\ = \;& \textstyle\int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x)\exp(-A_c[\Phi,\bar{\Phi}] \\ & -\textstyle\sum_{n,m}(\bar{\Psi}_n - \bar{\Phi}_n)\alpha_{n,m}(\Psi_m - \Phi_m)). \end{aligned} \tag{5}$$

Here $\alpha_{n,m}$ is a matrix acting on the multiplet, Ψ_n,

$$\alpha_{n,m} = \alpha\ \delta_{n,m}\begin{pmatrix} 0 & 0 & 1 \\ 0 & V & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6}$$

where V is some anti-symmetric matrix determined by a mass-term Lagrangian. This $\alpha_{n,m}$ term is a SUSY invariant mass term with a large mass $O(a^{-1})$. In the naïve sense, our effective action, $A_{\rm eff}$ is SUSY invariant but the transformation is modified as seen later.

We may define naïve SUSY on the lattice:

$$\begin{aligned} \delta^N_\epsilon \Phi_n &= \int f_n(x)\delta\Phi(x)\mathrm{d}x \\ &= Q_L(\epsilon,\bar{\epsilon};\overrightarrow{\nabla})\Phi_n \end{aligned} \tag{7}$$

$$\begin{aligned} \delta^N_\epsilon \bar{\Phi}_n &= \int f_n(x)\delta\bar{\Phi}(x)\mathrm{d}x \\ &= \bar{\Phi}_n\bar{Q}_L(\epsilon,\bar{\epsilon};\overleftarrow{\nabla}) \end{aligned} \tag{8}$$

This is just an interpretation from continuum SUSY to lattice SUSY, where a derivative operator in the continuum SUSY is replaced by a difference operator in the lattice SUSY using the relation $\partial_\mu f_n(x) = -\overrightarrow{\nabla}_\mu f_n(x)$. Although the explicit form of the difference operator depends on the block-spin function, it is possible to choose the reasonable function in the realization of lattice SUSY. Under this naïve lattice SUSY transformation, our effective action changes by

$$\begin{aligned} &\exp(-A_{\rm eff}[\Psi', \bar{\Psi}']) \\ = \;& \int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x)\exp(-A_c[\Phi,\bar{\Phi}] \\ & -(\bar{\Psi}' - \bar{\Phi})\alpha(\Psi' - \Phi)) \\ = \;& \int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x)\exp(-A_c[\Phi',\bar{\Phi}'] \\ & -(\bar{\Psi} - \bar{\Phi}')e^{\bar{Q}_L}\alpha e^{Q_L}(\Psi - \Phi')) \end{aligned}$$

where the lattice site and the spinor indices are both omitted. It is assumed that A_c is invariant under SUSY transformation in the continuum theory,

$$A_c[\Phi,\bar{\Phi}] = A_c[\Phi',\bar{\Phi}']. \tag{9}$$

Althogh the path-integral measure is naïvely unchanged

$$\mathcal{D}\Phi'\mathcal{D}\bar{\Phi}' = \mathcal{D}\Phi\mathcal{D}\bar{\Phi}, \tag{10}$$

we include the Jacobian factor in the following calculation and shall find the relation with the anomaly[13].

Let us derive a SUSY extension of the G-W relation for a free theory described by

$$A_{\rm eff}[\Psi,\bar{\Psi}] = \sum_{n,m}\bar{\Psi}_n S_{(n,m)}\Psi_m. \tag{11}$$

Under the naive SUSY, it transforms as

$$\begin{aligned} &\exp(-A_{\rm eff}[\Psi,\bar{\Psi}])\{1 - \bar{\Psi}(SQ_L + \bar{Q}_L S)\Psi\} \\ = \;& \{1 + \delta J + \mathrm{str}\ (Q_L + \bar{Q}_L) \\ & -\mathrm{str}\ (Q_L\alpha^{-1}S + S\alpha^{-1}\bar{Q}_L) \\ & -\bar{\Psi}S(Q_L\alpha^{-1} + \alpha^{-1}\bar{Q}_L)S\Psi\} \\ & \times\exp(-A_{\rm eff}[\Psi,\bar{\Psi}]), \end{aligned} \tag{12}$$

*Since we have two chiral multiplets, it is possible to construct a Dirac mass term.

where δJ comes from a Jacobian factor. So, we can get following two relations;

$$\begin{aligned}\delta J &= \text{str } (Q_L\alpha^{-1}S + S\alpha^{-1}\bar{Q}_L)\\ &\quad -\text{str } (Q_L+\bar{Q}_L),\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\bar{\Psi}(SQ_L+\bar{Q}_LS)\Psi\\ &\quad = \bar{\Psi}S(Q_L\alpha^{-1}+\alpha^{-1}\bar{Q}_L)S\Psi.\end{aligned} \tag{14}$$

These are SUSY extended GW relations. Note that the right hand side of Eq.(14) vanishes if the difference operator $(\overrightarrow{\triangledown})$ of $Q_L(\overrightarrow{\triangledown})$, $\bar{Q}_L(\overleftarrow{\triangledown})$ is symmetric. Therefore eq.(14) is divided into two cases according to the difference operator $(\overrightarrow{\triangledown})$. For a symmetric difference operator $\overrightarrow{\triangledown_s}$, the above relations suggest us to difine

$$q_s \equiv Q_L(\overrightarrow{\triangledown_s}) \tag{15}$$

$$\bar{q}_s \equiv \bar{Q}_L(\overleftarrow{\triangledown_s}) \tag{16}$$

under which we can show the invariance of our effective action:

$$\delta A_{\text{eff}} = \bar{\Psi}(Sq_s+\bar{q}_sS)\Psi = 0. \tag{17}$$

For a non-symmetric difference operator $\overrightarrow{\triangledown}$,

$$q \equiv Q_L(\overrightarrow{\triangledown}) - Q_L(\overrightarrow{\triangledown})\alpha^{-1}S \tag{18}$$

$$\bar{q} \equiv \bar{Q}_L(\overleftarrow{\triangledown}) - S\alpha^{-1}\bar{Q}_L(\overleftarrow{\triangledown}) \tag{19}$$

and

$$\delta A_{\text{eff}} = \bar{\Psi}(Sq+\bar{q}S)\Psi = 0. \tag{20}$$

This is a SUSY extension of Lüscher's symmetry[6].

For $U(1)_R$ charge Q_R, we can find the G-W relations similar to the SUSY case:

$$\begin{aligned}\delta J_R &= \text{str } (Q_R\alpha^{-1}S + S\alpha^{-1}\bar{Q}_R)\\ &\quad -\text{str } (Q_R+\bar{Q}_R),\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\bar{\Psi}(SQ_R+\bar{Q}_RS)\Psi\\ &\quad = \bar{\Psi}S(Q_R\alpha^{-1}+\alpha^{-1}\bar{Q}_R)S\Psi.\end{aligned} \tag{22}$$

The conserved '$U(1)_R$' charge [9],

$$q_R \equiv Q_R(1-\alpha^{-1}S), \tag{23}$$

can be also found as one of exact symmetries on the lattice.

3. An example: 2-Dimensional chiral multiplets

We consider two chiral-multiplets Φ_j, $j=1,2$, consisted of real scalars ϕ_j, auxiliary fields F_j and complex Weyl spinors χ_j. There are arranged in a complex multiplet $\Phi = (\phi_1+i\phi_2, F_1+iF_2; \chi_1+i\chi_2, \chi_1^*+i\chi_2^*)^{\mathrm{T}} \equiv (\phi, F; \chi, \bar{\chi})^{\mathrm{T}}$ and its congugate $\bar{\Phi} = (\phi_1-i\phi_2, F_1-iF_2; \chi_1-i\chi_2, \chi_1^*-i\chi_2^*)) \equiv (\phi^*, F^*; \bar{\chi}^\dagger, \chi^\dagger)$. We define N=1 SUSY transformation;

$$\begin{cases}\delta_\epsilon\phi = i(\epsilon^*\chi+\epsilon\bar{\chi})\\ \delta_\epsilon F = -2\epsilon\partial_{\bar{z}}\chi + 2\epsilon^*\partial_z\bar{\chi}\\ \delta_\epsilon\chi = -2\epsilon^*\partial_z\phi + i\epsilon F\\ \delta_\epsilon\bar{\chi} = -2\epsilon\partial_{\bar{z}}\phi - i\epsilon^*F.\end{cases} \tag{24}$$

We consider a SUSY-invariant massless Lagrangian:

$$\mathcal{L} = 2\partial_{\bar{z}}\phi^*\partial_z\phi - \frac{1}{2}F^*F + i(\chi^\dagger\partial_z\bar{\chi} + \bar{\chi}^\dagger\partial_{\bar{z}}\chi). \tag{25}$$

Now, we obtain the matrix V in Eq.(6),

$$V = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \tag{26}$$

from a mass-term in a SUSY-invariant Lagrangian,

$$\mathcal{L}_m = -\frac{m}{2}(F^*\phi + F\phi^* + \chi^\dagger\chi - \bar{\chi}^\dagger\bar{\chi}). \tag{27}$$

The naïve lattice SUSY takes of the matrix form

$$Q_L = \begin{pmatrix} 0 & 0 & i\epsilon^* & i\epsilon\\ 0 & 0 & -2\epsilon\triangledown_{\bar{z}} & +2\epsilon^*\triangledown_z\\ -2\epsilon^*\triangledown_z & i\epsilon & 0 & 0\\ -2\epsilon\triangledown_{\bar{z}} & -i\epsilon^* & 0 & 0\end{pmatrix}, \tag{28}$$

and

$$\bar{Q}_L = \begin{pmatrix} 0 & 0 & -2\epsilon^* \overleftarrow{\nabla}_z & -2\epsilon \overleftarrow{\nabla}_{\bar{z}} \\ 0 & 0 & i\epsilon & -i\epsilon^* \\ -i\epsilon^* & 2\epsilon \overleftarrow{\nabla}_{\bar{z}} & 0 & 0 \\ -i\epsilon & -2\epsilon^* \overleftarrow{\nabla}_z & 0 & 0 \end{pmatrix}. \tag{29}$$

It follows that

$$\alpha Q_L + \bar{Q}_L \alpha$$

$$= \alpha \begin{pmatrix} 0 & 0 & -2\epsilon(TD)_{\bar{z}} & 2\epsilon^*(TD)_z \\ 0 & 0 & 0 & 0 \\ 2\epsilon(TD)_{\bar{z}} & 0 & 0 & 0 \\ -2\epsilon^*(TD)_{\bar{z}} & 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

where $(TD)_z$ means a total derevative $\overrightarrow{\nabla}_z + \overleftarrow{\nabla}_z$. This result Eq.(30) is mentioned in Eq.(14). Therefore, if one uses the symmetric difference operator, lattice SUSY is exactly same as naïve one.

Let us derive SUSY extended GW relations. We denote the lattice effective action A_{eff}:

$$A_{\text{eff}}[\Psi, \bar{\Psi}] = \sum_{n,m} \bar{\Psi}_n S_{(n,m)} \Psi_m, \tag{31}$$

$$S = \begin{pmatrix} D_{\text{B}} & 0 \\ 0 & D_{\text{F}} \end{pmatrix}, \tag{32}$$

where D_{B} is a bosonic kinetic part, and D_{F} is a fermionic one. From Eq.(14), we obtain in our approach not only the original GW relation but also the relation among fermions and bosons:

$$\sigma_3 D_{\text{F}} + D_{\text{F}} \sigma_3 = \alpha^{-1} D_{\text{F}} \sigma_3 D_{\text{F}}, \tag{33}$$

$$\Sigma_3 D_{\text{B}} + D_{\text{B}} \Sigma_3 = \alpha^{-1} D_{\text{B}} \Sigma_3 D_{\text{B}}, \tag{34}$$

$$-2 D_{\text{B}}^{F^* F} \overrightarrow{\nabla}_{\bar{z}} + i D_{\text{F}}^{\bar{\chi}^\dagger \chi} = \alpha^{-1} D_{\text{B}}^{F^* F} (TD)_{\bar{z}} D_{\text{F}}^{\chi^\dagger \chi}, \tag{35}$$

$$i D_{\text{B}}^{\phi^* \phi} - 2 \overleftarrow{\nabla}_z D_{\text{F}}^{\bar{\chi}^\dagger \chi} = \alpha^{-1} D_{\text{B}}^{\phi^* F} (TD)_z D_{\text{F}}^{\chi^\dagger \chi}, \tag{36}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the superscript $\chi^\dagger \chi$ of $D_{\text{F}}^{\chi^\dagger \chi}$ denotes the $(\chi^\dagger, \chi)$ component of D_{F}. Eq.(33) for fermions is just the original GW relation for chiral symmetry, and due to SUSY Eq.(34) for bosons was obtained as the corresponding relation.[9] Therefore SUSY can live with $U(1)_R$ symmetry on the lattice. Also Eqs.(35),(36) represent the supersymmetric ralation among fermions and bosons. Once we find the fermion kinetic term $D_F^{\bar{\chi}^\dagger \chi}$, it is easy to express the total action explicitly from these relations.

REFERENCES

1. H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185** (1981) 20; ERRATUM **B195** (1982) 541; **B193** (1981) 173.
2. D. Kaplan, Phys. Lett. **B288** (1992) 342.
3. R. Narayanan and H. Neuberger, Phys. Rev. Lett. **71** (1993) 325; Phys. Lett. **B302** (1993) 62; Nucl. Phys. **B443** (1995) 305.
4. H. Neuberger, Phys. Lett. **B417** (1998) 141; **B427** (1998) 353.
5. P. Ginsparg and K. Wilson, Phys. Rev. **D25** (1982) 2649.
6. M. Lüscher, Phys. Lett. **B428** (1998) 342.
7. N. Sakai and M. Sakamoto, Nucl. Phys. **B229** (1983) 173.
8. S. Nojiri, Prog. Theor. Phys. **74** (1985) 819.
9. T. Aoyama and Y. Kikukawa, Phys. Rev. **D59** (1999) 054507.
10. W. Bietenholz, Mod. Phys. Lett. **A14** (1999) 51.
11. H. So and N. Ukita, Phys. Lett. **B457** (1999) 314.
12. M. Koseki, H. So and N. Ukita, hep-lat/0011009.
13. K. Fujikawa, Phys. Rev. **D21** (1980) 2848; ERRATUM **D22** (1980) 1499.